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Maximal Inequalities and Lebesgue's Differentiation Theorem for Best Approximant by Constant over Balls[1]

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For $f \in L_p(\mathbb{R}^n)$, with $1 \leqslant p < \infty$, $\varepsilon > 0$ and $x \in \mathbb{R}^n$ we denote by $T^\varepsilon(f)(x)$ the set of every best constant approximant to f in the ball $B(x, \varepsilon)$. In this paper we extend the operators T_p^ε to the space $L_{p-1}(\mathbb{R}^n) + L_\infty(\mathbb{R}^n)$, where L_0 is the set of every measurable functions finite almost everywhere. Moreover we consider the maximal operators associated to the operators T_p^ε and we prove maximal inequalities for them. As a consequence of these inequalities we obtain a generalization of Lebesgue's Differentiation Theorem. © 2001 Academic Press

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1. INTRODUCTION AND NOTATION

In this paper we consider a problem related to best local approximation. The notion may be stated as follows. Let f: $\mathbb{R}^n \to \mathbb{R}$ be a function in a normed linear space X with norm $\|\cdot\|$. Let V denote a subset of X. Let $B(x, \varepsilon)$ denote a net of sets containing x with diameters shrinking to 0 as $\varepsilon \to 0$. For each $\varepsilon > 0$ suppose that we have $f_\varepsilon \in V$ which minimizes $\|(f-g)\,\chi_{B(x,\varepsilon)}\|$ for $g \in V$, where $\chi_{B(x,\varepsilon)}$ is the characteristic function of $B(x, \varepsilon)$. If $f_\varepsilon \to f_x \in V$ then f_x is said to be the best local approximant of f at x. In [1] Chui, Diamond and Raphael proved that if f have $m+1$ derivatives at x and the subspace $V \subset C^{m+1}(\mathbb{R}^n)$ is uniquely interpolating at x of order m then the best local approximant of f at x from V is the unique $f_x \in V$ whose derivatives up to order m match those of f at x.

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Clearly the condition $V \subset X$ can be substituted by $V\chi_{B(x,\varepsilon)} \subset X$ for every $\varepsilon > 0$. We observe that if $X = L_2(\mathbb{R}^n)$, V is the set of constant functions and $B(x, \varepsilon)$ is the ball of center in x and radius ε then

$$f_\varepsilon(x) = \frac{1}{m(B(x, \varepsilon))} \int_{B(x, \varepsilon)} f(t)\, dt.$$

As a consequence of the previous result we have that $f_\varepsilon(x) \to f(x)$ for $f \in C^1(\mathbb{R}^n)$ and for every $x \in \mathbb{R}^n$. But it is well known that a more adequate version of this fact is given by the Lebesgue's Differentiation Theorem, which says that $f_\varepsilon(x) \to f(x)$ a.e. for every locally integrable function f over $\mathbb{R}^n$. This theorem is related to some inequalities satisfied by the Hardy–Littlewood maximal function, see [3].

In the present work we extend the maximal function of Hardy–Littlewood and the Lebesgue's Differentiation Theorem using best approximation by constants over balls in the $L_p(\mathbb{R}^n)$ spaces with $1 \leqslant p < \infty$.

In [2] Landers and Rogge have also considered problems related with maximal inequalities and almost everywhere convergence of best approximant. They studied these questions in L_p-spaces with $1 < p < \infty$. In particular they extend the operator of best approximation from L_p to L_{p-1}. In this paper we analyze the case $p = 1$.

Throughout this paper $B(x, \varepsilon)$ denotes the ball in $\mathbb{R}^n$ with center in x and radius ε. For $1 \leqslant p < \infty$, $f \in L_p(\mathbb{R}^n)$ and $\varepsilon > 0$ we define $T_p^\varepsilon(f)(x)$ as the set of all constants a minimizing the expression

$$\int_{B(x, \varepsilon)} |f(t) - a|^p\, dt.$$

It is well known that $T_p^\varepsilon(f)(x) \neq \emptyset$ for every $f \in L_p(\mathbb{R}^n)$. Moreover if $1 < p < \infty$ then the set $T_p^\varepsilon(f)(x)$ has an unique element.

For our purposes we define the following maximal function over an open set $\Omega \subset \mathbb{R}^n$.

$$T^*_{\Omega, p}(f)(x) = \sup_{\varepsilon > 0} \{|a| : a \in T_p^\varepsilon(f)(x) \text{ and } B(x, \varepsilon) \subset \Omega\}, \tag{1}$$

where $1 \leqslant p < \infty$.

2. MAXIMAL INEQUALITIES

For short we put L_p instead of $L_p(\mathbb{R}^n)$. Since for $f \in L_p + L_\infty$, $\int_{B(x,\varepsilon)} |f(x)|^p\, dx < \infty$ for any ball, we have that the operator T_p^ε is defined for all functions in L_p+L_∞. Moreover the following property holds:

(P1) If g is constant on $B(x,\varepsilon)$ then $T_p^\varepsilon(f+g)(x)=T_p^\varepsilon(f)(x)+g(x)$.

Next we shall see that T_p^ε admits a natural extension. For $p>1$ we extend T_p^ε to $L_{p-1}+L_\infty$, so that it satisfies (P1). Notice that for $1<p<2$ the L_{p-1} space is not a normed space, however we still maintain the notation $\|f\|_p$ for $(\int |f(x)|^p\,dx)^{1/p}$.

For $f\in L_{p-1}+L_\infty$, $\varepsilon>0$ and $x\in\mathbb{R}^n$ we consider the following function

$$F(a):=\int_{B(x,\varepsilon)} |f(t)-a|^{p-1}\operatorname{sgn}(f(t)-a)\,dt, \qquad a\in\mathbb{R}.$$

By the characterization theorem of best approximant we get $F(T^\varepsilon(f)(x))=0$ for $f\in L_p$.

Now we see that for every $f\in L_{p-1}+L_\infty$ there exists an unique $a\in\mathbb{R}$ such that $F(a)=0$, so we define $T_p^\varepsilon(f)(x)=a$. In fact, as the function $\Phi(x)=|x|^{p-1}\operatorname{sgn}(x)$ is continuous and strictly increasing we have that the function F is continuous and strictly decreasing. Furthermore it satisfies that $\lim_{a\to+\infty}F(a)=-\infty$ and $\lim_{a\to-\infty}F(a)=+\infty$. Then there exists an unique number a such that $F(a)=0$.

If $p=1$ we extend the operator T_1^ε to the space L_0 of all measurable functions which are finite almost everywhere. In this case we consider the distribution function $\lambda(a)=m(\{t\in B(x,\varepsilon):\ f(t)>a\})$. For every $f\in L_0$, $x\in\mathbb{R}^n$ and $\varepsilon>0$ we define

$$U^\varepsilon(f)(x):=\sup\{t\colon \lambda(t)\geqslant m(B(x,\varepsilon))/2\}$$

$$L^\varepsilon(f)(x):=\inf\{t\colon \lambda(t)\leqslant m(B(x,\varepsilon))/2\}.$$

For $f\in L_1$ it is well known, see [4, p. 199], that $T_1^\varepsilon(f)(x)=[L^\varepsilon(f)(x), U^\varepsilon(f)(x)]$. So that we extend the operator T_1^ε to the space L_0 by mean of this equality.

Clearly the extended operator T_p^ε satisfies (P1). Also we extend the maximal function $T^*_{\Omega,p}$ using (1). It is easy to show that the operator T_p^ε, with $1\leqslant p<\infty$, satisfies:

(P2) for every $\alpha\in\mathbb{R}$, $T_p^\varepsilon(\alpha f)=\alpha T_p^\varepsilon(f)$

(P3) T_p^ε is monotone, i.e., for all f, g with $f\leqslant g$, a.e. we have $T_p^\varepsilon(f)\leqslant T_p^\varepsilon(g)$.

From (P2) and (P3) follows that

(P4) $|T_p^\varepsilon(f)|\leqslant T_p^\varepsilon(|f|)$.

Since the function $T_p^{\varepsilon}(f)$ is single valued for $p>1$, we obtain from Lebesgue's Dominated Convergence Theorem that $T_p^{\varepsilon}(f)$ is continuous. Hence the maximal function $T_{p,\Omega}^*$ is lower semi-continuous, in particular is a measurable function. If $p=1$ then $U^{\varepsilon}(f)$ $(L^{\varepsilon}(f))$ is upper (lower) semi-continuous. This affirmation follows easily from the fact that the function $g(x) := m(\{t \in B(x, \varepsilon) : f(t) > a\})$ is continuous for every a. So $U^{\varepsilon}(f)$ and $L^{\varepsilon}(f)$ are measurable functions. However we can not prove that the maximal function $T_1^*(f)$ is measurable. As a consequence we will use the outer measure m^* in some results. Now we prove maximal inequalities for the operator $T_{\Omega,p}^*$.

THEOREM 2.1. *Let f be a measurable function over Ω and let $1 \leqslant p < \infty$ then*

(a) *There exist constants A_{p-1} and k_{p-1} such that*

$$m^*(\{x \in \Omega : T_{\Omega,p}^*(f)(x) > \lambda\}) \leqslant \frac{A_{p-1}}{\lambda^{p-1}} \int_{\{|f| > k_{p-1}\lambda\}} |f(t)|^{p-1} \, dt$$

for every $f \in L_{p-1}$ and $\lambda > 0$.

(b) *Let $p>1$. For $p-1 < p' \leqslant \infty$ there exists a constant $A_{p'}$ such that*

$$\|T_{\Omega,p}^*(f)\|_{p'} \leqslant A_{p'} \|f\|_{p'}.$$

Proof. Let $M_\lambda := \{x \in \Omega : T_{\Omega,p}^*(f)(x) > \lambda\}$. In order to prove part (a) we need to consider two cases.

Case 1. $1 < p < \infty$. In this case the set M_λ is measurable. For $x \in M_\lambda$ there is $\varepsilon = \varepsilon_x > 0$ such that $|T_p^{\varepsilon}(f)(x)| > \lambda$ and $B(x, \varepsilon) \subset \Omega$. Now we see that there exists a constant C such that

$$m(B(x, \varepsilon)) \leqslant \frac{C}{\lambda^{p-1}} \int_{B(x,\varepsilon)} |f(t)|^{p-1} \, dt. \tag{2}$$

In fact, it is easy to show that there are constants B_p and C_p with $B_p \leqslant 1 \leqslant C_p$ such that

$$B_p(a^{p-1} + b^{p-1}) \leqslant (a+b)^{p-1} \leqslant C_p(a^{p-1} + b^{p-1}), \qquad a \geqslant 0, \quad b \geqslant 0. \tag{3}$$

For simplicity we put $N := B(x, \varepsilon) \cap \{f > T_p^{\varepsilon}(f)(x)\}$ and $L := B(x, \varepsilon) \cap \{f \leqslant T_p^{\varepsilon}(f)(x)\}$. Then

$$\begin{aligned} m(B(x,\varepsilon))\,\lambda^{p-1} &\leqslant m(B(x,\varepsilon))\,|T_p^\varepsilon(f)(x)|^{p-1} \\ &= \int_N |T_p^\varepsilon(f)(x)|^{p-1}\,dt + \int_L |T_p^\varepsilon(f)(x)|^{p-1}\,dt \\ &\leqslant C_p \int_N |T_p^\varepsilon(f)(x)|^{p-1}\,dt + C_p \int_L |f(t)|^{p-1}\,dt \\ &\quad + C_p \int_L |T_p^\varepsilon(f)(x) - f(t)|^{p-1}\,dt \\ &\leqslant C \int_{B(x,\varepsilon)} |f(t)|^{p-1}\,dt, \end{aligned}$$

where $C = \frac{C_p}{B_p}$. The last inequality follows from the equality

$$\int_L |T_p^\varepsilon(f)(x) - f(t)|^{p-1}\,dt = \int_N |T_p^\varepsilon(f)(x) - f(t)|^{p-1}\,dt$$

and from (3).

Thus we have proved that the diameters of the balls $B(x, \varepsilon_x)$ are bounded. Now, by [3, Lemma 1.6], we can select from this family of balls a sequence $B(x_i, \varepsilon_{x_i})$ of balls which are mutually disjoint and such that

$$\sum_i m(B(x_i, \varepsilon_{x_i})) \geqslant D m(M_\lambda),$$

for some constant $D > 0$. Therefore, from (2) we obtain

$$\begin{aligned} m(M_\lambda) &\leqslant \frac{C}{D\lambda^{p-1}} \sum_i \int_{B(x_i, \varepsilon_{x_i})} |f(t)|^{p-1}\,dt \\ &\leqslant \frac{C}{D\lambda^{p-1}} \int_\Omega |f(t)|^{p-1}\,dt. \end{aligned}$$

Now we define $f_1(x) = f(x)$ if $|f(x)| > \lambda/2$ and $f_1(x) = 0$ otherwise. Then $|f(x)| \leqslant |f_1(x)| + \lambda/2$. From (i), (iii) and (1) follows that $T^*_{\Omega,p}(f)(x) \leqslant T^*_{\Omega,p}(f_1)(x) + \lambda/2$. Therefore

$$m(M_\lambda) \leqslant \frac{2^{p-1}C}{D\lambda^{p-1}} \int_\Omega |f_1(x)|\,dx = \frac{A_{p-1}}{\lambda^{p-1}} \int_{\{|f| > k_{p-1}\lambda\}} |f(x)|^{p-1}\,dx,$$

where $A_{p-1} = 2^{p-1}C/D$ and $k_{p-1} = 1/2$. This prove the case 1.

Case 2. $p=1$. If $m(\{t\in\Omega : |f(t)|>\lambda\})=\infty$ then the inequality in (a) is trivial. So that we can suppose $m(\{t\in\Omega : |f(t)|>\lambda\})<\infty$. If $x\in M_\lambda$ then there exist $\varepsilon=\varepsilon_x$ and $a\in T_1^\varepsilon(f)(x)$ such that $|a|>\lambda$ and $B(x,\varepsilon)\subset\Omega$. We shall prove that

$$m(B(x,\varepsilon))\leqslant 2m(\{t\in B(x,\varepsilon) : |f(t)|>\lambda\}). \tag{4}$$

In fact, if $a>0$ then $\lambda<U^\varepsilon(f)(x)$. Hence $m(\{t\in B(x,\varepsilon) : |f(t)|>\lambda\})\geqslant m(\{t\in B(x,\varepsilon) : f(t)>\lambda\})\geqslant m(B(x,\varepsilon))/2$. Suppose $a<0$ and let $a<s<-\lambda$. We have that

$$\begin{aligned} m(\{t\in B(x,\varepsilon) : |f(t)|>\lambda\}) &\geqslant m(\{t\in B(x,\varepsilon) : f(t)<-\lambda\})\\ &= m(B(x,\varepsilon))-m(\{t\in B(x,\varepsilon) : f(t)\geqslant -\lambda\})\\ &\geqslant m(B(x,\varepsilon))-m(\{t\in B(x,\varepsilon) : f(t)>s\})\\ &\geqslant m(B(x,\varepsilon))/2. \end{aligned}$$

The last inequality follows from $L^\varepsilon(f)(x)\leqslant a$. Then (4) follows.

Since $m(\{t\in\Omega : |f(t)|>\lambda\})<\infty$ we obtain from (4) that the diameters of the balls $B(x,\varepsilon_x)$ are bounded. Let $G:=\bigcup_{x\in M_\lambda} B(x,\varepsilon)$. The set G is open. Again, an application of [3, Lemma 1.6] give us a countable subcollection of balls mutually disjoint and $D>0$ such that

$$Dm^*(M_\lambda)\leqslant Dm(G)\leqslant\sum_i m(B(x_i,\varepsilon_i))\leqslant 2m(\{t\in\Omega : |f(t)|>\lambda\}).$$

This conclude the proof of case two if we take $k_0=1$ and $A_0=2/D$.

Now we prove part (b). Since $|f|\leqslant\|f\|_\infty$, a.e. we have that (P3) and (P4) imply $T_p^\varepsilon(f)\leqslant\|f\|_\infty$ a.e. Hence we obtain (b) for $p'=\infty$ with $A_\infty=1$. Using a similar argument as given in [3, p. 7] and using the properties (P1) and (P3) we obtain (b) for the case $p-1<p'$. ∎

3. LEBESGUE'S DIFFERENTIATION THEOREM

In the present section we shall prove a generalization of the "Lebesgue's Differentiation Theorem" for the operators $T_p^\varepsilon f$. First we need to prove the following theorem. By $\mathcal{M}=\mathcal{M}(\Omega)$ we denote the set of all measurable functions over Ω.

THEOREM 3.1. *Let* $\{T^\varepsilon\}_{\varepsilon>0}$ *be a family of operators, not necessarily linear, where* $T^\varepsilon\colon L_p(\Omega)\to\mathcal{M}$ *for some* $0\leqslant p<\infty$. *We consider the maximal* *function* $T^*(f)(x) := \sup_{\varepsilon>0} |T^\varepsilon(f)(x)|$. *Suppose that* T^* *satisfies the following weak inequality,*

$$m^*(\{x \in \Omega : T^*(f)(x) > \lambda\}) \leqslant \frac{A}{\lambda^p} \int_{\{|f| > k\lambda\}} |f(x)|^p \, dx, \tag{5}$$

for some constants $A, k > 0$. *Moreover assume that there exists a set* $\mathscr{D} \subset L_p(\Omega)$ *with the following properties*:

(A1) *For every* $\lambda > 0$, $f \in L_p(\Omega)$ *and* $g \in \mathscr{D}$

$$m^*(\{x \in \Omega : \limsup_{\varepsilon \to 0} |T^\varepsilon(f-g)(x) - T^\varepsilon(f)(x) + T^\varepsilon(g)(x)| > \lambda\}) = 0.$$

(A2) *If* $f \in L_p(\Omega)$ *then for every* $\varepsilon > 0$ *and* $\lambda > 0$ *there exists* $g \in \mathscr{D}$ *such that*

$$\int_{\{|f-g| > \lambda\}} |(f-g)(t)|^p < \varepsilon.$$

(A3) $\lim_{\varepsilon \to 0} T^\varepsilon(g)(x) = g(x)$ *a.e. for every* $g \in \mathscr{D}$.

Then we have that $\lim_{\varepsilon \to 0} T^\varepsilon(f)(x) = f(x)$ *a.e. for every* $f \in L_p(\Omega)$.

Proof. It is enough to prove that $m^*(\Omega_0) = 0$ for every $\lambda > 0$, where

$$\Omega_0 := \{x \in \Omega : \limsup_{\varepsilon \to 0} |T^\varepsilon(f)(x) - f(x)| > \lambda\}$$

For $g \in \mathscr{D}$ we have that $\Omega_0 \subset \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$ where

$$\Omega_1 := \{x \in \Omega : \limsup_{\varepsilon \to 0} |T^\varepsilon(f-g)(x) - T^\varepsilon(f)(x) + T^\varepsilon(g)(x)| > \lambda/4\}$$

$$\Omega_2 := \{x \in \Omega : \limsup_{\varepsilon \to 0} |T^\varepsilon(g)(x) - g(x)| > \lambda/4\}$$

$$\Omega_3 := \{x \in \Omega : |g(x) - f(x)| > \lambda/4\}$$

$$\Omega_4 := \{x \in \Omega : \limsup_{\varepsilon \to 0} |T^\varepsilon(f-g)(x)| > \lambda/4\}.$$

By (A1) and (A3) we have that $m^*(\Omega_1) = m^*(\Omega_2) = 0$. Let $\delta > 0$ be arbitrary. From (A2) we get $g \in \mathscr{D}$ with $\int_{\{|f-g| > \alpha\}} |(f-g)(x)|^p \, dx < \delta$, where $\alpha := \min\{\lambda/4, k\lambda/4\}$. Therefore we see that $m(\Omega_3) < \delta/\alpha^p$. Moreover inequality (5) implies that

$$\begin{aligned} m^*(\Omega_4) &\leqslant m^*(\{x \in \Omega : T^*(f-g)(x) > \lambda/4\}) \\ &\leqslant \frac{4^p A}{\lambda^p} \int_{\{|f-g| > k\lambda/4\}} |(f-g)(x)|^p \, dx \\ &\leqslant \frac{4^p A}{\lambda^p} \delta. \end{aligned}$$

Since the number δ is arbitrary we deduce that $m^*(\Omega_0) = 0$. ∎

COROLLARY 3.2. *For* $1 < p < \infty$ *and* $f \in L_{p-1}(\mathbb{R}^n) + L_\infty(\mathbb{R}^n)$ *we have that*

$$\lim_{\varepsilon \to 0} T_p^\varepsilon(f)(x) = f(x) \text{ a.e.}$$

If $p = 1$ *then*

$$\lim_{\varepsilon \to 0} U^\varepsilon(f)(x) = \lim_{\varepsilon \to 0} L^\varepsilon(f)(x) = f(x) \text{ a.e.}$$

for every $f \in L_0(\mathbb{R}^n)$.

Proof. It is sufficient to prove that the convergence a.e. holds in all open set Ω of finite measure. Let Ω be an open set with $m(\Omega) < \infty$. We observe that if $f \in L_{p-1}(\mathbb{R}^n) + L_\infty(\mathbb{R}^n)$ then $f \in L_{p-1}(\Omega)$. Hence from Theorem 2.1 we have that T_p^ε, U^ε and L^ε satisfy (5). Let $\mathscr{D}$ the set of all simple functions which are continuous almost everywhere, i.e. $g \in \mathscr{D}$ if $g = \sum_{i=1}^n a_i \chi_{A_i}$ with $m(\partial A_i) = 0$, for every $i = 1, ..., n$, where ∂A denote the boundary of the set A. As a consequence of property (P1) we have that for $T^\varepsilon = T_p^\varepsilon$, U^ε or L^ε and $g = \sum_{i=1}^n a_i \chi_{A_i} \in \mathscr{D}$

$$\{x \in \Omega : \limsup_{\varepsilon \to 0} |T^\varepsilon(f-g)(x) - T^\varepsilon(f)(x) + T^\varepsilon(g)(x)| > \lambda\} \subset \bigcup_{i=1}^n \partial A_i.$$

so that we have (A1) for every $1 \leqslant p < \infty$. The property (A2) is a consequence of the well known properties of density of the simple functions (observe that (A2) is not true if $m(\Omega) = \infty$ and $p = 1$). Moreover for $g \in \mathscr{D}$ it is easy to prove that (A3) holds for the operators T_p^ε, U^ε and L^ε. ∎

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